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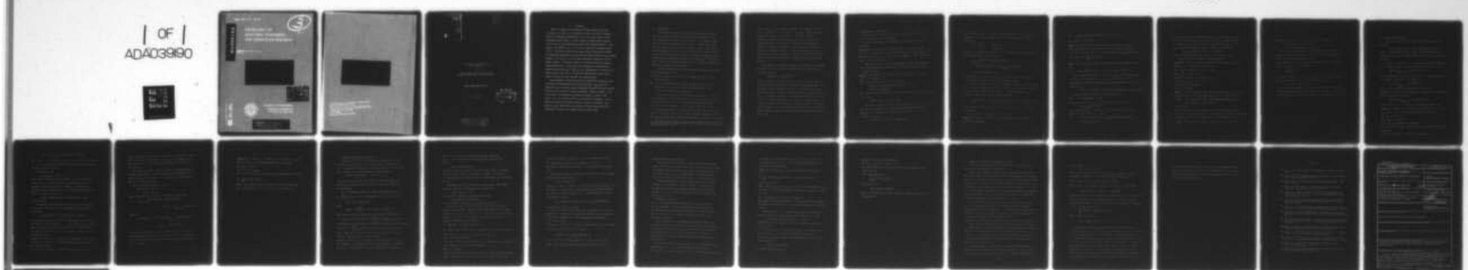

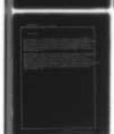

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Convergence of Image Measures
and Integrals

by

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Under AFOSR Grant 74-2627

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(See form 1473)

Abstract

This is a study of the convergence of image measures $\mu_n f_n^{-1}(A) = \mu_n\{x: f_n(x) \in A\}$, where μ_n is a measure on X which is finite on compacts, and f_n is a measurable mapping from X to Y, two locally compact second countable Hausdorff spaces. A basic result for image measures asserts that if the μ_n's are probability measures which converge weakly to μ, and $f_n(x_n) \to f(x)$ whenever $x_n \to x$ for μ a.e. x, then $\mu_n f_n^{-1}$ converges weakly to μf^{-1}. We present a similar result and a partial converse for general μ_n, where vague convergence is used instead of weak convergence (μ_n converges vaguely to μ if $\int f\, d\mu_n \to \int f\, d\mu$ for each continuous function f on X with compact support). Its proof is based on a Fatou-like lemma for vaguely converging measures. We also study the convergence in distribution of random image measures $\xi_n \phi_n^{-1}$ where ξ_n is a random measure on X and ϕ_n is a random function from X to Y. We show how these measures can be used to analyze thinnings of point processes and random measures.

The convergence of integrals $\int f_n\, d\mu_n$ is essentially equivalent to the convergence of image measures, since $\int_X f_n(x)\, d\mu_n(x) = \int_R t\, d\mu_n f_n^{-1}(t)$. Using this idea, we present several convergence theorems for these integrals when the μ_n's are weakly or vaguely convergent. These are similar to the result that if μ-integrable f_n converge in μ-measure to some f, then $\int f_n\, d\mu \to \int f\, d\mu$ if and only if the f_n are uniformly μ-integrable. We also extend our integral convergence theorems to mixtures of measures $\nu_n(A) = \int k_n(x,A)\, d\mu_n(x)$, which arise in the study of extreme order statistics of exchangeable variables, and randomly selected partial sums.

1. Introduction

Let X denote a locally compact second countable Hausdorff space, let $\mathcal{X}$ be the Borel σ-algebra on X generated by its topology, and let $b\mathcal{X}$ be the bounded (i.e. relatively compact) sets in $\mathcal{X}$. We denote by $M_b(X)$ the set of finite (nonnegative) measures on X. A sequence μ_n in $M_b(X)$ converges weakly to μ, written $\mu_n \overset{w}{\to} \mu$ in $M_b(X)$, if $\int f d\mu_n \to \int f d\mu$ for any bounded continuous function f on X; see [1] and [3]. We let M(X) denote the set of measures on X that are finite on compact sets (Radon Measures). A sequence μ_n in M(X) converges vaguely to μ, written $\mu_n \overset{v}{\to} \mu$ in M(X), if $\int f d\mu_n \to \int f d\mu$ for any continuous function f on X with compact support; see [2], [6] or [8] for the basics on this.

The subtle difference between weak and vague convergence can be seen by the following statements.

(1) $\mu_n \overset{w}{\to} \mu$ in $M_b(X)$ if and only if $\mu_n \overset{v}{\to} \mu$ in $M_b(X)$ and $\mu_n(X) \to \mu(X)$ (see [8, p. 74]).

(2) $\mu_n \overset{v}{\to} \mu$ in M(X) if and only if $\mu_n \overset{w}{\to} \mu$ in $M_b(K)$ (here the μ's are restricted to K) for each compact K in X whose boundary ∂K has μ-measure zero. (This is easy to prove.)

In this article we study the convergence of image measures and some related integrals. More specifically, let F(X,Y) be the set of measurable functions from X to Y, where Y is a second countable locally compact Hausdorff space. By an image of $\mu \in M(X)$ under a function $f \in F(X,Y)$ we mean the measure $\mu f^{-1}(A) = \mu\{x: f(X) \in A\}$ for $A \in \mathcal{Y}$. The weak convergence of images $P_n f_n^{-1}$, where P_n are weakly convergent probability measures, is studied in [3] and [14]. In particular, Theorem 5.5 in [3] (due to H. Rubin)

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asserts that if $P_n \overset{w}{\to} P$ in $M_b(X)$, and $f_n(x_n) \to f(x)$ whenever $x_n \to x$ for P a.e. x, then $P_n f_n^{-1} \overset{w}{\to} Pf^{-1}$ in $M_b(Y)$. We present a vague convergence analogue of this as well as a converse. Its proof is based on a Fatou-like lemma for vaguely convergent measures. We also present a random version of it dealing with the convergence in distribution of random image measures $\xi_n \phi_n^{-1}$ where ξ_n is a random measure on the non-negative real numbers R_+ and ϕ_n is a random function from R_+ to Y. We discuss how such random image measures can be used for analyzing thinned point processes and random measures as studied in [7], [8], [10], [11] and [12].

The convergence of integrals $\int f_n d\mu_n$, for $f_n \in F(X,R)$, is essentially equivalent to the convergence of the image measures $\mu_n f_n^{-1}$, because, by a change of variable,

$$\int_X f_n(x)d\mu_n(x) = \int_R t d\mu_n f_n^{-1}(t).$$

Using this idea, we obtain (in Section 3) several convergence theorems for these integrals when the μ_n's are weakly or vaguely convergent. These are similar to the well-known result that if μ-integrable $f_n \in F(X,R_+)$ converge in μ-measure to some $f \in F(X,R_+)$ then $\int f_n d\mu \to \int f d\mu$ if and only if the f_n are uniformly μ-integrable. We pursue this theme further in Section 4 where we study the convergence of measures $\lambda_n(A) = \int k_n(x,A)d\mu_n(x)$. Our results here contain the key Theorem 2.1 in [5] on mixtures of probability measures, which is used for analyzing extreme order statistics of exchangeable variables or random sized samples, and for analyzing randomly selected partial sums.

2. Convergence of Image Measures

In this section we consider the vague convergence of image measures $\mu_n f_n^{-1}$. We begin with a few preliminaries.

The type of convergence on the functions f_n that we shall assume is as follows.

Definition 2.1. Let f and f_n be in $F(X,Y)$ and let μ and μ_n be in $M(X)$ for $n \geq 1$. We say that f_n converges continuously to f a.e. μ_n if $f_n(x_n) \to f(x)$ whenever $x_n \to x$ for $x_n \in B_n \in \mathcal{X}$, $x \in B \in \mathcal{X}$ and $\mu_n(B_n^c) = 0 = \mu(B^c)$. We denote this by $f_n \overset{c}{\to} f$ in $F(X,Y)$ a.e. μ_n.

The Theorem 5.5 in [3], on the convergence of images of probabilities which we mentioned above, readily extends to images of arbitrary measures in $M_b(X)$ as follows.

Theorem 2.2. Suppose $\mu_n \overset{w}{\to} \mu$ in $M_b(X)$ and $f_n \overset{c}{\to} f$ in $F(X,Y)$ a.e. μ_n. Then $\mu_n f_n^{-1} \overset{w}{\to} \mu f^{-1}$ in $M_b(Y)$.

Proof. The assertion $\mu_n f_n^{-1} \overset{w}{\to} \mu f^{-1}$ *is equivalent to*

$$\mu f^{-1}(G) \leq \liminf_n \mu_n f_n^{-1}(G) \text{ for all open } G \in \mathcal{Y}, \text{ and}$$

$$\mu f^{-1}(Y) = \lim_n \mu_n f_n^{-1}(Y).$$

The inequality follows as in the proof of [3, Theorem 5.5]. The second statement follows since

$$\mu_n f_n^{-1}(Y) = \mu_n(X) \to \mu(X) = \mu f^{-1}(Y).$$

Our aim is to prove an analogue of the preceeding for vague convergence. The last preliminary we need for this is the following Fatou-like result.

Theorem 2.3. If either $\mu_n \overset{w}{\to} \mu$ in $M_b(X)$ or $\mu_n \overset{v}{\to} \mu$ in $M(X)$, and $f_n \overset{c}{\to} f$ in $F(X,R_+)$ a.e. μ_n, then

$$\int |f(x)|\, d\mu(x) \leq \liminf_n \int |f_n(x)|\, d\mu_n(x).$$

Proof. Suppose that $\mu_n \overset{w}{\to} \mu$ in $M_b(X)$. Then by Theorem 2.2 we have $\mu_n f_n^{-1} \overset{w}{\to} \mu f^{-1}$ in $M_b(R)$. Let $a > 0$ be such that $-a$ and a are continuity point of the measure μf^{-1}, and let

$$g_a(t) = \begin{cases} |t| & \text{for } |t| \leq a \\ 0 & \text{otherwise.} \end{cases}$$

Clearly g_a is bounded and continuous a.e. μf^{-1}. Then, using a standard change of variable formula for integrals, we have

$$\int_{\{|f(x)| \leq a\}} |f(x)|\, d\mu(x) = \int_R g_a(t)\, d\mu f^{-1}(t)$$

$$= \lim_n \int_R g_a(t)\, d\mu_n f_n^{-1}(t) = \lim_n \int_{\{|f_n(x)| \leq a\}} |f_n(x)|\, d\mu_n(x)$$

$$\leq \liminf_n \int |f_n(x)|\, d\mu_n(x).$$

An application of the monotone convergence theorem to the first integral, as $a \to \infty$ through continuity points of μf^{-1}, yields the assertion.

Now consider the case in which $\mu_n \overset{v}{\to} \mu$ in $M(X)$. Let $K_1 \subset K_2 \subset \ldots$ be compact sets in X such that $\bigcup_{i=1}^{\infty} K_i = X$ and $\mu(\partial K_i) = 0$ for all i. The latter is possible by Lemma 2.6 below. For each K_i we have $\mu_n \overset{w}{\to} \mu$ in $M_b(K_i)$ (here the μ's are restricted to K_i). Then by the preceeding paragraph,

$$\int_{K_i} |f(x)|\, d\mu(x) \leq \liminf_n \int_{K_i} |f_n(x)|\, d\mu_n(x) \leq \liminf_n \int_X |f_n(x)|\, d\mu_n(x).$$

An application of the monotone convergence theorem to the first integral yields the assertion.

We now come to the main result of this section.

Theorem 2.4. Suppose $\mu_n \overset{v}{\to} \mu$ in $M(X)$, $f_n \overset{c}{\to} f$ in $F(X,Y)$ a.e. μ_n, the μf^{-1} and $\mu_n f_n^{-1}$ are in $M(Y)$, and

(1) $$\inf_{B\in b\mathcal{X}} \limsup_n \mu_n(f_n^{-1}(K)\cap B^c) = 0 \quad \text{for all compact } K \in \mathcal{Y}.$$

Then $\mu_n f_n^{-1} \overset{v}{\to} \mu f^{-1}$ in $M(Y)$.

Remark. Condition (1) involving both the μ's and f's may be difficult to verify in some cases. A sufficient condition for (1) involving only the f's is this: For any compact $K \in \mathcal{Y}$, there is an N such that $\bigcup_{n=N}^{\infty} f_n^{-1}(K) \in b\mathcal{X}$. To see this let $B = \bigcup_{n=N}^{\infty} f_n^{-1}(K)$ in (1).

Proof. We shall prove the assertion by establishing the equivalent statement that

(2) $$\lim_n \mu_n f_n^{-1}(A) = \mu f^{-1}(A) \quad \text{for all } A \in b\mathcal{Y} \text{ with } \mu f^{-1}(\partial A) = 0.$$

Pick an $A\in b\mathcal{Y}$ with $\mu f^{-1}(\partial A) = 0$. Let $g(\cdot) = 1_A(\cdot)$, the indicator function of A. Since $f_n \overset{c}{\to} f$ in $F(X,Y)$ *a.e.* μ_n, *and* g *is continuous a.e.* μf^{-1}, then it follows by an elementary argument that $g\circ f_n \overset{c}{\to} g\circ f$ in $F(X,R_+)$ a.e. μ_n. Then by Theorem 2.3,

$$\mu f^{-1}(A) = \int g\circ f(x)\, d\mu(x) \le \liminf_n \int g\circ f_n(x)\, d\mu_n(x) = \liminf_n \mu_n f_n^{-1}(A).$$

We shall complete the proof of (2) by showing that

(3) $$\limsup_n \mu_n f_n^{-1}(A) \le \mu f^{-1}(A).$$

To this end, fix an $\varepsilon > 0$, and choose a compact B in X such that

$$\limsup_n \mu_n(f_n^{-1}(\bar{A})\cap B^c) < \varepsilon.$$

This is possible by (1). In addition, take B to be such that $\mu(\partial B) = 0$. This is possible by Lemma 2.6. Clearly

(4) $$\limsup_n \mu_n f_n^{-1}(A) = \limsup_n (\mu_n(f_n^{-1}(A)\cap B) + \mu_n(f_n^{-1}(A)\cap B^c))$$

$$\le \limsup_n \tilde{\mu}_n \tilde{f}_n^{-1}(A) + \varepsilon$$

where $\tilde{\mu}_n$ and $\tilde{f}_n$ are the restrictions of μ_n and f_n, respectively, to the set B. Under our assumptions it follows that $\tilde{\mu}_n \overset{w}{\to} \tilde{\mu}$ in $M_b(B)$ and $\tilde{f}_n \overset{c}{\to} \tilde{f}$ in F(B,Y) a.e. $\tilde{\mu}_n$, and so by Theorem 2.2, $\tilde{\mu}_n\tilde{f}_n^{-1} \overset{w}{\to} \tilde{\mu}\tilde{f}^{-1}$ in $M_b(Y)$. The latter implies that $\tilde{\mu}_n\tilde{f}_n^{-1}(A) \to \tilde{\mu}\tilde{f}^{-1}(A)$. Using this in (4) yields

$$\limsup_n \mu_n f_n^{-1}(A) \le \tilde{\mu}\tilde{f}^{-1}(A) + \varepsilon \le \mu f^{-1}(A) + \varepsilon.$$

Since ε was chosen arbitrarily, this proves (3), and we are done.

The next result is a converse to the last theorem.

Theorem 2.5. Let μ_n be in M(X) and f_n be in F(X,Y) such that $\mu_n f_n^{-1}$ is in M(Y) for $n \geq 1$. Suppose the following hold.

(i) $\mu_n f_n^{-1} \overset{v}{\to} \mu f^{-1}$ in M(Y) for some $\mu \in M(X)$.

(ii) $f_n \overset{c}{\to} f$ in F(X,Y) a.e. μ_n.

(iii) $f^{-1}(\mathcal{Y})$ contains the open sets of X.

(iv) $\sup_n \mu_n(B) < \infty$ for all $B \in bX$.

(v) The $\mu_n f_n^{-1}$ satisfy (1).

Then $\mu_n \overset{v}{\to} \mu$ in M(X).

Remark. A weak convergence version of the above is as follows. If μ_n is in $M_b(X)$, conditions (ii) and (iii) hold, and $\mu_n f_n^{-1} \overset{w}{\to} \mu f^{-1}$ in $M_b(Y)$, then $\mu_n \overset{w}{\to} \mu$ in $M_b(X)$. This is proved similarly. In this context (iv) holds, since

$$\mu_n(X) = \mu_n f_n^{-1}(Y) \to \mu f^{-1}(Y) = \mu(X)$$

implies (iv).)

Proof. Condition (iv) is equivalent to the μ_n being vaguely relatively compact, [8, p. 94]. Consequently, any subsequence of μ_n contains a subsequence $\mu_{n'}$ such that $\mu_{n'} \overset{v}{\to}$ some λ in M(X) as $n' \to \infty$. By Theorem 2.4 we have $\mu_{n'} f_{n'}^{-1} \overset{v}{\to} \lambda f^{-1}$ in M(Y). From this and (i) it follows that $\lambda f^{-1} = \mu f^{-1}$.

For any open set A in X, let $B \varepsilon Y$ be such that $A = f^{-1}(B)$. Then

$$\lambda(A) = \lambda f^{-1}(B) = \mu f^{-1}(B) = \mu(A).$$

Thus $\lambda = \mu$, and so the original sequence $\mu_n \xrightarrow{v} \mu$.

We end this section with the following lemma, which we used above.

Lemma 2.6. If μ is in $M(X)$ and K is a compact set in X, then there is a compact $U \varepsilon X$ containing K with $\mu(\partial U) = 0$.

Proof. For each $x \varepsilon K$, let V_x be a compact neighborhood of x. Since K is compact, there are $x_1, \ldots, x_n$ such that

$$K \subset \bigcup_{i=1}^{n} V_{x_i} \equiv V.$$

By Urysohn's lemma there is a continuous function f from X to the interval $[0,1]$ such that $f(x) = 1$ if $x \varepsilon K$ and $f(x) = 0$ if $x \varepsilon V^c$.

$$\text{Let } K_\varepsilon = \{x : f(x) \geq \varepsilon\} \quad \text{for } \varepsilon > 0.$$

Each K_ε is closed and contained in V and so it is compact. Also because f is continuous, then $\partial K_\varepsilon \subset \{x \varepsilon V : f(x) = \varepsilon\}$. It follows, since μ is finite on V, that there is an ε' such that $\mu(\partial K_{\varepsilon'}) = 0$. This completes the proof.

3. Convergence of Integrals

In this section, we study the convergence of the integrals and measures

$$\mu_n f_n = \int f_n(x) d\mu_n(x) \quad \text{and} \quad f_n\mu_n(A) = \int_A f_n(x) d\mu_n(x) \qquad \text{for } A \in X,$$

where $f_n \in F(X,R)$ and $\mu_n \in M(X)$. We show how their convergence is related to the uniform μ_n-integrability of the f's, which is defined as follows.

Functions $f_n \in F(X,R)$ $(n \geq 1)$ are said to be uniformly μ-integrable, for $\mu \in M(X)$, if there is an $h \in F(X,(0,\infty))$ such that $\mu h < \infty$ and

$$\lim_{a\to\infty} \sup_n \int_{\{|f_n(x)| > ah(x)\}} |f_n(x)| d\mu(x) = 0.$$

(See [2, Theorem 2.12.7] concerning σ-finite μ's.) In particular, the f_n are uniformly μ-integrable for $\mu \in M_b(X)$ if

$$\lim_{a\to\infty} \sup_n \int_{\{|f_n(x)| > a\}} |f_n(x)| d\mu(x) = 0.$$

We shall use the following generalization of the latter.

Definition 3.1. Let f_n be in $F(X,R)$ and μ_n be in $M_b(X)$. We say that the f_n are uniformly μ_n-integrable if

$$\lim_{a\to\infty} \lim\sup_n \int_{\{|f_n(x)| > a\}} |f_n(x)|\, d\mu_n(x) = 0.$$

(This definition can be extended to μ_n in $M(X)$ in an obvious way.)

Theorem 3.2. Suppose $\mu_n \xrightarrow{w} \mu$ in $M_b(X)$; $f_n \xrightarrow{c} f$ in $F(X,R_+)$ a.e. μ_n; and $\mu_n f_n < \infty$ for $n \geq 1$. The following statements are equivalent.

(i) $\mu_n f_n \to \mu f$ and $\mu f < \infty$.

(ii) $f_n\mu_n \xrightarrow{w} f\mu$ in $M_b(X)$.

(iii) The f_n are uniformly μ_n-integrable.

(iv) The functions $g_n(t) = \mu_n f_n^{-1}(t,\infty)$ $(t \geq 0)$ are uniformly Lebesgue-integrable.

Proof. If (iii) holds, then there is an M such that

$$\mu_n f_n \leq M + \int_{\{f_n(x) \leq a\}} f_n(x)\, d\mu_n(x) \leq M + a\mu_n(X).$$

Since $\mu_n(X) \to \mu(X)$, then $\limsup_n \mu_n f_n < \infty$, and so by Theorem 2.3, we have $\mu f < \infty$. To finish proving that (i) is equivalent to (iii), just parrot the proof of [3, Theorem 5.4].

If (ii) holds, then

$$\mu_n f_n = f_n \mu_n(X) \to f\mu(X) = \mu f < \infty,$$

which is (i). Now suppose that (iii) holds. We will show that (ii) follows. Let g be a bounded, continuous nonnegative function on X. Clearly the functions $f_n(\cdot)g(\cdot)$ are uniformly μ_n-integrable and $f_n(\cdot)g(\cdot) \overset{c}{\to} f(\cdot)g(\cdot)$ in $F(X,R_+)$ a.e. μ_n. Using the established fact that (iii) implies (i) we obtain

$$f_n\mu_n(g) = \int g(x) f_n(x)\, d\mu_n(x) \to \int g(x) f(x)\, d\mu(x) = f\mu(g).$$

This yields (ii).

We finish the proof by showing that (i) and (iv) are equivalent. By a change of variable and a well-known expression for expectations we can write

$$\mu_n f_n = \int_0^\infty t\, d\mu_n f_n^{-1}(t) = \int_0^\infty \mu_n f_n^{-1}(t,\infty) dt = \int_0^\infty g_n(t) dt.$$

From Theorem 2.2 we have $\mu_n f_n^{-1} \overset{w}{\to} \mu f^{-1}$. Then $g_n(t) \to g(t) \equiv \mu f^{-1}(t,\infty)$ for all but a countable number of t's. It follows, by the classical convergence theorem for uniformly integrable functions, [2, Cor. 2.12.5], that $\int g_n(t)dt \to \int g(t)dt$ if and only if (iv) holds. This proves that (i) is equivalent to (iv).

Theorem 3.3. Suppose $\mu_n \overset{v}{\to} \mu$ in $M(X)$; $f_n \overset{c}{\to} f$ in $F(X,R_+)$ a.e. μ_n; and the $f\mu$ and $f_n\mu_n$ are in $M(X)$. The following statements are equivalent.

(i) $f_n\mu_n \overset{v}{\to} f\mu$ in $M(X)$.

(ii) The f_n are uniformly μ_n-integrable on each compact set in X.

Proof. Suppose (i) holds. Let K be a compact set in X, and pick a compact set K' containing K with $f\mu(\partial K') = 0$. Then $f_n\mu_n \xrightarrow{w} f\mu$ in $M_b(K')$, and so by Theorem 3.2 the f_n are uniformly μ_n-integrable on K' and hence on K. This proves (ii)

Now suppose (ii) holds. Let g be a continuous nonnegative function on X with compact support. From the definition of weak convergence, it is clear that $g\mu_n \xrightarrow{w} g\mu$ in $M_b(X)$. Let $\lambda_n = (g\mu_n)f_n^{-1}$. By Theorem 2.2 we have $\lambda_n \xrightarrow{w} \lambda \equiv (g\mu)f^{-1}$ in $M_b(R_+)$.

Let a be a continuity point of λ and let

$$h(t) = \begin{cases} |t| & \text{for } 0 \le t \le a \\ 0 & \text{for } t > a. \end{cases}$$

Using a change of variable formula for integrals, it follows that

$$\int_{\{f_n(x) \le a\}} f_n(x)g(x)d\mu_n(x) = \int_{R_+} h(t)d(g\mu_n)f^{-1}(t)$$

$$= \lambda_n h \to \lambda h = \int_{\{f(x) \le a\}} f(x)g(x)\, d\mu(x).$$

Therefore,

$$\limsup_n |f_n\mu_n g - f\mu g| \le \limsup_n \left| \int_{\{f_n(x) > a\}} f_n(x)g(x)d\mu_n(x) \right.$$

$$\left. - \int_{\{f(x) > a\}} f(x)g(x)d\mu(x) \right|.$$

As in the proof of Theorem 3.2 one can show that $f\mu g < \infty$. This and the uniform μ_n-integrability of the f_n insures that as $a \to \infty$, the right hand side of the above inequality converges to zero. Thus $f_n\mu_n \xrightarrow{v} f\mu$ and the proof is complete.

Corollary 3.4. Suppose $\mu_n \xrightarrow{v} \mu$ in $M(X)$; $f_n \xrightarrow{c} f$ in $F(X,R)$ a.e. μ_n; and $\mu_n f_n < \infty$ for $n \geq 1$. The following statements are equivalent

(i) $\mu_n f_n \to \mu f$ and $\mu f < \infty$.

(ii) $f_n \mu_n \xrightarrow{w} f\mu$ in $M_b(X)$.

(iii) The f_n are uniformly μ_n-integrable on each compact set in X and

$$(1) \quad \inf_{B \varepsilon bX} \limsup_n f_n \mu_n (B^c) = 0.$$

Proof. This follows directly from Theorems 3.2 and 3.3 and the fact [8, p. 95] that $f_n \mu_n \xrightarrow{w} f\mu$ if and only if $f_n \mu_n \xrightarrow{v} f\mu$ and (1) hold.

4. Convergence of Mixtures of Measures

A kernel k from X to Y is a mapping $(x,A) \to k(x,A)$ from $X \times \mathcal{Y}$ to R_+ such that $k(\cdot,A) \in F(X,R_+)$ for each A, and $k(x,\cdot) \in M(Y)$ for each $x \in X$. We let $K(X,Y)$ be the set of all kernels from X to Y. For $\mu \in M(X)$ and $k \in K(X,Y)$ we let μk be the measure in $M(X\times Y)$ defined by

$$\mu k(A\times B) = \int_A k(x,B)d\mu(x) \qquad \text{for } A \in \mathcal{X},\ B \in \mathcal{Y}.$$

In this section, we study the convergence of such measures $\mu_n k_n$. Our results are extensions of those in [5] on the convergence of mixtures of probabilities.

As an illustration, consider the randomly selected partial sums $Z_n = \sum_{k=1}^{N_n} X_{n,k}$, where the N's are independent of the X's. We can write

$$P(Z_n \in B) = \int k_n(x,B)d\mu_n(x)$$

where

$$k_n(x,B) = P\Big(\sum_{k=1}^{[xa_n]} X_{n,k} \in B\Big), \text{ and } \mu_n(A) = P(N_n/a_n \in A).$$

Our first result (Theorem 4.1) implies that if $k_n(x,\cdot) \xrightarrow{w} k(x,\cdot)$ for all $x \in R$ and $\mu_n \xrightarrow{w} \mu$ in $M(R)$ (that is, the $\sum_{k=1}^{[xa_n]} X_{n,k}$ and N_n/a_n converge in distribution), then Z_n converges in distribution. Our second result (Theorem 4.2) implies that if $\sum_{k=1}^{[xa_n]} X_{n,k}$ and Z_n converge in distribution, then so does N_n/a_n. Other applications to extreme value statistics of exchangeable events or random size samples are given in [5].

Theorem 4.1 Suppose $\mu_n \xrightarrow{w} \mu$ in $M_b(X)$, and k and k_n in $K(X,Y)$ are such that $\mu_n k_n(X\times Y) < \infty$ and $k_n(x_n,\cdot) \xrightarrow{w} k(x,\cdot)$ in $M_b(Y)$ whenever $x_n \to x$ where $x_n \in B_n$, $x \in B$, and $\mu_n(B_n^c) = 0 = \mu(B^c)$. Then $\mu_n k_n \xrightarrow{w} \mu k$ in $M_b(X\times Y)$ if and only if the functions $k_n(\cdot,Y)$ are uniformly μ_n- integrable.

Proof. Let f be a bounded continuous nonnegative function on X×Y and let

$$h_n(x) = \int f(x,y)k_n(x,dy).$$

For any $x_n \to x$ with $x_n \in B_n$ and $x \in B$, the functions $f(x_n,\cdot)$ are bounded, and clearly $f(x_n,\cdot) \overset{c}{\to} f(x,\cdot)$ in $F(Y,R_+)$ a.e. $k_n(x_n,\cdot)$. Then Theorem 3.2 yields $h_n(x_n) \to h(x) = \int f(x,y)k(x,dy)$. In other words $h_n \overset{c}{\to} h$ in $F(X,R_+)$ a.e. μ_n.

Now suppose the $k_n(\cdot,Y)$ are uniformly μ_n-integrable. Then clearly the h_n are uniformly μ_n-integrable and so by Theorem 3.2

$$\mu_n k_n f = \mu_n h_n \to \mu h = \mu k f.$$

Thus $\mu_n k_n \overset{w}{\to} \mu k$. Conversely, if the latter holds then

$$\int k_n(x,Y)d\mu_n(x) \to \int k(x,Y)d\mu(x)$$

and so by Theorem 3.2 the $k_n(\cdot,Y)$ are uniformly μ_n-integrable.

In the next result, we use the statement that $k \in K(X,Y)$ identifies mixtures in a set $\Gamma \subset M(X)$. This means that if $\int k(x,\cdot)d\lambda(x) = \int k(x,\cdot)d\mu(x)$ in $M(Y)$ for any λ and μ in Γ, then $\lambda = \mu$. See [5] and the references therein.

Theorem 4.2. Let μ_n and μ be in $M_b(X)$, and k_n and k be in $K(X,Y)$ for $n \geq 1$. Suppose the following conditions hold.

(i) $\sup_n \mu_n(X) < \infty$ and $\inf\limits_{B \in bX} \sup_n \mu_n(B^c) = 0$.

(ii) $\int k_n(x,\cdot)d\mu_n(x) \overset{w}{\to} \int k(x,\cdot)d\mu(x)$ in $M_b(Y)$ where μ is in some set $\Gamma \subset M(X)$ which contains the vague limits of subsequences of $\mu_n (n \geq 1)$.

(iii) k identifies mixtures in Γ.

(iv) $k_n(x_n,\cdot) \overset{w}{\to} k(x,\cdot)$ in $M_b(Y)$ whenever $x_n \to x$, where $x_n \in B_n$ and $\mu_n(B_n{}^c) = 0$; and $x \in B$ where B is closed and $\liminf_n \mu_n(B^c) = 0$.

Then $\mu_n \overset{w}{\to} \mu$ in $M_b(X)$.

Proof. Condition (i) is equivalent to the $\mu_n (n \geq 1)$ being weakly relatively compact. Then for any subset of integers one can choose another subset

N such that $\mu_n \xrightarrow{w}$ some λ in $M_b(X)$ as $n \to \infty$ in N. Suppose we knew that

(1) $\int k_n(x,\cdot)d\mu_n(x) \xrightarrow{w} \int k(x,\cdot)d\lambda(x)$ in $M_b(Y)$ as $n \to \infty$ in N.

Then from (ii) and (iii) we would have $\mu = \lambda$, which proves the assertion $\mu_n \xrightarrow{w} \mu$.

To complete the proof, we only need to verify (1). Let f be a bounded continuous function on Y and let

$$h_n(x) = \int f(y)k_n(x,dy).$$

For any $x_n \to x$ with $x_n \in B_n$ and $x \in B$, it follows by (iv) that $h_n(x_n) \to h(x)$ and $\lambda(B^c) = 0$. Also the $h_n (n \in N)$ are uniformly μ_n-integrable by Theorem 3.2, since (ii) implies that $\mu_n h_n \to \mu h < \infty$. Applying Theorem 3.2 to the μ_n and $h_n (n \in N)$ we get

$$\int_X \int_Y f(y)k_n(x,dy)d\mu_n(x) = \mu_n h_n \to \lambda h = \int_X \int_Y f(y)k(x,dy)d\lambda(x).$$

This proves (1), and so we are done.

We now show how the above results apply to convolutions. For this we assume that the space X is also a group with addition as the operation. We let $\lambda*\mu$ be the convolution of λ and μ in $M_b(X)$ which is defined by $\lambda*\mu(A) = \int \lambda(A-x)d\mu(x)$.

Corollary 4.3.(a) If $\lambda_n \xrightarrow{w} \lambda$ and $\mu_n \xrightarrow{w} \mu$ in $M_b(X)$, then $\lambda_n * \mu_n \xrightarrow{w} \lambda*\mu$ in $M_b(X)$.

(b) Suppose $\lambda_n * \mu_n \xrightarrow{w} \lambda*\mu$ and $\lambda_n \xrightarrow{w} \lambda$ in $M_b(X)$,

$$\sup_n \mu_n(X) < \infty \text{ and } \inf_{B \in bX} \sup_n \mu_n(B^c) = 0,$$

and $\lambda*\mu = \lambda*\nu$ for $\mu,\nu \in M_b(X)$ implies $\mu = \nu$. Then $\mu_n \xrightarrow{w} \mu$ in $M_b(X)$.

Proof. These assertions follow directly from Theorems 4.1 and 4.2.

5. Random Image Measures and Integrals

The above results can readily be extended to describe the convergence in distribution of random measures $\xi_n\phi_n^{-1}$, $\xi_n*\eta_n$, and $\int\gamma_n(x,\cdot)\, d\xi_n(x)$, and integrals $\int\phi_n(x)\, d\xi_n(x)$, where ξ_n and η_n are random measures, ϕ_n are random functions and γ_n are random kernels. We illustrate this here by presenting random versions of Theorem 2.3 and Corollary 3.4.

We begin by defining the type of random functions that we shall consider. Let $G \equiv G(X,Y)$ be a subset of $F(X,Y)$. We assume that G is endowed with a separable, metrizable topology with the following property: If $f_n \to f$ in G and f is continuous at x, then $f_n(x_n) \to f(x)$ whenever $x_n \to x$ in X.

Examples of such G are:

(a) $G = \{$continuous functions on $[0,1]\}$ with the uniform topology

(b) $G = \{f \in F(R,R)$: f is right continuous and nonincreasing$\}$ with the relativized Skorohod topology [3, Chapter 3], or the weak topology.

(c) $G = \{f \in F(X,Y)$: f is continuous$\}$ with the topology of uniform convergence on compacta.

A random element of G is defined to be a measurable mapping from a probability space to $(G,\mathcal{G})$, where $\mathcal{G}$ is the smallest σ-field containing the topology of G. Similarly, a random measure on X (a random element of $M(X)$) is defined to be a measurable mapping from a probability space to $(M(X), \mathcal{M})$ where $\mathcal{M}$ is the smallest σ-field containing the vague topology of $M(X)$.

For our first result, we assume $\xi, \xi_1, \xi_2, \ldots$ are random measures on X and $\phi, \phi_1, \phi_2, \ldots$ are random elements of $G(X,Y)$ such that $\xi\phi^{-1}, \xi_1\phi_1^{-1}, \ldots$ are random measures on Y. We assume that $\xi_1, \phi_1, \xi_2, \phi_2, \ldots$

are defined on a common probability space. We let D_f be the set of discontinuity points of $f \in G$. We let $\to_{\mathcal{D}}$ denote weak convergence in distribution of random elements.

Theorem 5.1. If $(\xi_n, \phi_n) \to_{\mathcal{D}} (\xi, \phi)$, and with probability one $\xi(D_\phi) = 0$ and

(1) $\inf_{B \in b\mathcal{X}} \limsup_n \xi_n(\phi_n^{-1}(K) \cap B^c) = 0$ for any compact $K \in \mathcal{X}$,

then $\xi_n \phi_n^{-1} \to_{\mathcal{D}} \xi\phi^{-1}$

Proof. Since $(\xi_n, \phi_n) \to_{\mathcal{D}} (\xi, \phi)$, then there exist random elements $\xi^*, \xi_1^*, \ldots$ and $\phi^*, \phi_1^*, \ldots$ on a common probability space such that $(\xi_n^*, \phi_n^*) \to (\xi^*, \phi^*)$ a.s., and

(2) $(\xi_n^*, \phi_n^*) =_{\mathcal{D}} (\xi_n, \phi_n)$ and $(\xi^*, \phi^*) =_{\mathcal{D}} (\xi, \phi)$.

This follows by the Skorohod-Dudley a.s. representation of convergence in distribution of random elements of a separable metric space, see [4], [9] and [13].

Clearly $\xi^*(D_{\phi^*}) = 0$ a.s. and the (ξ_n^*, ϕ_n^*) satisfy condition (1). Moreover, the nature of the topology on G and $\xi^*(D_{\phi^*}) = 0$ a.s. imply

$$\xi^*\{x : \phi_n^*(x_n) \not\to \phi^*(x) \text{ for some } x_n \to x\} = 0 \text{ a.s.}$$

Then Theorem 2.3 yields $\xi_n^* \phi_n^{*-1} \to \xi^* \phi^{*-1}$ a.s. From this and (2) we have $\xi_n \phi_n^{-1} \to_{\mathcal{D}} \xi\phi^{-1}$.

For the next result we assume that $\xi, \xi_1, \xi_2, \ldots$ are random measures on X, and $\phi, \phi_1, \phi_2, \ldots$ are random elements of $G(X, R_+)$ such that $\xi_1, \phi_1, \xi_2, \phi_2, \ldots$ are defined on a common probability space and the $\phi_n \xi_n$ (and $\phi\xi$) defined by

$$\phi_n \xi_n(A) = \int_A \phi_n(x) d\xi_n(x)$$

are finite random measures on X.

Theorem 5.2. Suppose the following hold.

(i) $(\xi_n, \phi_n) \to_{\mathcal{D}} (\xi, \phi)$ and $\xi(D_\phi) = 0$ a.s.

(ii) With probability one, the ϕ_n are uniformly ξ_n-integrable on each compact set in X.

(iii) With probability one,

$$\inf_{B \in b\mathcal{X}} \lim\sup_n \phi_n \xi_n (B^c) = 0.$$

Then $\phi_n \xi_n \to_{\mathcal{D}} \phi\xi$ and

$$\int \phi_n(x) d\xi_n(x) \to_{\mathcal{D}} \int \phi(x) d\xi(x).$$

Proof. This follows from Corollary 3.4 and an argument like that in the preceeding proof.

6. Thinning of Point Processes and Random Measures

In this section we show how the above results on image measures can be used in analyzing thinnings of random measures.

In [11] it was shown that many thinnings of point processes, as in [7], [8], [10] and [12], can be described as follows. Suppose that mass is randomly placed on R_+ according to a random measure ζ. (The ζ is a point process if with probability one $\zeta(A)$ is integer-valued for each $A\varepsilon bX$.) Assume that this mass is thinned by another random measure η on R_+, with $\eta(t) = \eta([0,t]) \leq t$ a.s. for all $t \geq 0$, such that the mass $\zeta(t) = \zeta([0,t])$ in the interval $[0,t]$ is replaced by an amount $\eta(\zeta(t))$. In other words, the retained mass (the thinned random measure) is represented by the composition $\eta o \zeta$ of η and ζ.

One result in this setting is the following. For this we assume that $\eta_n o\zeta$ is a thinned random measure as described above where the thinning measure η_n depends on n and $\eta_n(R_+) = \infty$ a.s. We also let η be another random measure on η and let c and a_n be in R_+ with $a_n \to \infty$. By $\mu o a$ for $\mu\varepsilon M(R_+)$ and $a\varepsilon R_+$ we mean the measure defined by $\mu o a[0,t] = \mu(at)$.

Theorem 6.1. If $t^{-1}\zeta(t) \to_{\mathcal{D}} c$, then $\eta_n\ o\zeta o a_n \to_{\mathcal{D}} \eta o c$ if and only if $\eta_n o a_n \to_{\mathcal{D}} \eta$.

This is proved for point processes in [7] and for random measures in [11]. It contains the first result in thinning, see Renyi (1956), which is as follows. Suppose ζ is a renewal process whose interpoint distances have mean α. If each point of ζ is independently retained with probability p_n, where $p_n \to 0$, and ξ_n is the thinned process, then $\xi_n \circ p_n^{-1}$ converges in distribution to a Poisson process with intensity α^{-1}.

The proof in [11] of the above theorem is based on the following result which describes the continuity of the composition operator. This is proved directly in [10], but here we note that it is a corollary of our Theorem 2.4.

For this we let

$$\hat{\mu}(t) = \inf \{s : \mu(s) \geq t\}$$

be the left-continuous inverse of $\mu(t) = \mu[0,t]$ for $\mu \in M(R_+)$ with $\mu(R_+) = \infty$.

Corollary 6.2. Suppose $\lambda_n \overset{v}{\to} \lambda$ and $\mu_n \overset{v}{\to} \mu$ in $M(R_+)$, where $\mu_n(R_+) = \infty$, $\mu(\{0\}) = 0$, and $\lambda(D_{\hat{\mu}}) = 0$. Then $\lambda_n o \mu_n \overset{v}{\to} \lambda o \mu$ in $M(R_+)$.

Proof. Clearly $0 \leq s \leq \mu_n(t)$ if and only if $0 \leq \hat{\mu}_n(s) \leq t$. Consequently,

$$\lambda_n o \mu_n(t) = \lambda_n \{s \geq 0 : \hat{\mu}_n(s) \leq t\} = \lambda_n \hat{\mu}_n^{-1}[0,t].$$

Similar to the proof of [11, Theorem 2.1] it follows from $\mu_n \overset{v}{\to} \mu$ and $\mu(\{0\}) = 0$, that $\hat{\mu}_n(t) \to \hat{\mu}(t)$ for each continuity point t of μ. Since these μ's are nondecreasing and $\lambda(D_{\hat{\mu}}) = 0$, then one can easily show that $\hat{\mu}_n \overset{c}{\to} \hat{\mu}$ in $F(R_+, R_+)$ a.e. λ_n. Furthermore, for each t

$$\bigcup_{n=1}^{\infty} \hat{\mu}_n^{-1}[0,t] = \bigcup_{n=1}^{\infty} [0,\mu_n(t)] \in bR_+.$$

Thus by Theorem 2.4 we have

$$\lambda_n o \mu_n = \lambda_n \hat{\mu}_n^{-1} \overset{w}{\to} \lambda \hat{\mu}^{-1} = \lambda o \mu.$$

We began this section by describing a thinning of a random measure ζ on R_+ by another random measure η such that the thinned measure is the composition $\eta o \zeta$. An obvious question is, how can this type of "ordered" thinning on R_+ be characterized on a general space X, which may not be totally ordered? Since $\eta o \zeta = \eta \zeta^{-1}$ (as we saw in the last proof), then the following characterization by image measures is rather natural. Think of R_+ as a slab of mass that is deposited on X according to a random function ϕ from R_+ to X such that the t-th bit of R_+ is deposited at the location $\phi(t)$. Thin the deposited mass by a random measure η on R_+ such that the mass $\phi^{-1}(A)$ deposited in a set $A \in X$

is replaced by an amount $\eta(\phi^{-1}(A))$. That is, the thinned measure is the image measure $\eta\phi^{-1}$. Our Theorems 2.2 - 2.6 and Theorem 5.1 can be used to describe the convergence of such thinnings $\eta_n\phi_n^{-1}$ where the η's and ϕ's depend on converging parameters.

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This is a study of the convergence of image measures $\mu_n f_n^{-1}(A) = \mu_n\{x : f_n(x) \in A\}$, where μ_n is a measure on X which is finite on compacts, and f_n is a measurable mapping from X to Y, two locally compact second countable Hausdorff spaces. A basic result for image measures asserts that if the μ_n's are probability measures which converge weakly to μ, and $f_n(x_n) \to f(x)$ whenever $x_n \to x$ for μ a.e. x, then $\mu_n f_n^{-1}$ converges weakly to μf^{-1}. We present a similar result, and a partial converse, for general μn, where

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20. Abstract

vague convergence is used instead of weak convergence (μ_n converges vaguely to μ if $\int f\, d\mu_n \to f\, d\mu$ for each continuous function f on X with compact support). Its proof is based on a Fatou-like lemma for vaguely converging measures. We also study the convergence in distribution of random image measures $\xi_n \phi_n^{-1}$ where ξ_n is a random measure on X and ϕ_n is a random function from X to Y. We show how these measures can be used to analyze thinnings of point processes and random measur s.

The convergence of integrals $\int f_n d\mu_n$ is essentially equivalent to the convergence of image measures, since $\int_X f_n(x)\, d\mu_n(x) = \int_R t\, d\mu_n f_n^{-1}(t)$. Using this idea, we present several convergence theorems for these integrals when the μ_n's are weakly or vaguely convergent. These are similar to the result that if μ-integrable f_n converge in μ-measure to some f, then $\int f_n\, d\mu \to \int f\, d\mu$ if and only if the f_n are uniformly μ-integrable. We also extend our integral convergence theorems to mixtures of measures $\nu_n(A) = \int k_n(x,A)\, d\mu_n(x)$, which arise in the sutdy of extreme order statistics of exchangeable variables, and randomly selected partial sums.